MATHEMATICAL PSYCHICS

AN ESSAY ON THE

APPLICATION OF MATHEMATICS TO

THE MORAL SCIENCES

BY

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INTRODUCTORY

DESCRIPTION OF

CONTENTS.

Mathematical Psychics may be divided into two parts—Theoretical and Applied.

In the First Part (1) it is attempted to illustrate the possibility of Mathematical reasoning without *numerical* data (pp. 1–7); without more precise data than are afforded by estimates of *quantity of pleasure* (pp. 7–9). (2) An analogy is suggested between the *Principles of Greatest Happiness*, Utilitarian or Egoistic, which constitute the first principles of Ethics and Economics, and those *Principles of Maximum Energy* which are among the highest generalisations of Physics, and in virtue of which mathematical reasoning is applicable to physical phenomena quite as complex as human life (pp. 9–15).

The Calculus of Pleasure (Part II.) may be divided into two species—the Economical and the Utilitarian; the principle of division suggesting an addition to Mr. Sidgwick's 'ethical methods' (p. 16).

The first species of *Calculus* (if so ambitious a title may for brevity be applied to short studies in Mathematical Economics) is developed from certain *Definitions* of leading conceptions, in particular of those connected with *Competition* (pp. 17–19). Then (α) a mathematical theory of *Contract unqualified by Competition* is given (pp. 20–30). (β) A mathematical theory of *Contract determined by Competition in a perfect Market* is given, or at least promised (pp. 30–33, and pp. 38–42). Reference is made to other mathematical theories of Market, and to Mr. Sidgwick's recent article on the 'Wages-Fund' (pp. 32, 33, and Appendix V.) (γ) attention is concentrated on the question—*What is a perfect Market?* It is argued that Market is imperfect, *Contract is indeterminate* in the following cases:—

(I.) When the number of competitors is limited (pp. 37, 39).

(II.) In a certain similar case likely to occur in contracts for *personal service* (pp. 42, 46).

(I. and II.) When the *articles* of contract are not perfectly divisible (p. 42, 46).

(III.) In case of *Combination*, Unionism; in which case it is submitted that (in general and abstractly speaking) *unionists stand to gain* in senses contradicted or ignored by distinguished economists (pp. 44, 47, 48).

(IV.) In a certain case similar to the last, and likely to occur in *Co-operative Association* (pp. 45, 49).

The *indeterminateness* likely from these causes to affect *Commercial Contracts*, and certainly affecting all sorts of *Political Contracts*, appears to postulate a *principle of arbitration* (pp. 50–52).

It is argued from mathematical considerations that *the basis of arbitration between contractors is the greatest possible utility of all concerned;* the Utilitarian first principle, which can of course afford only a general

direction—yet, as employed by Bentham's school, has afforded *some* direction in practical affairs (pp. 53–56).

The Economical thus leads up to the Utilitarian species of Hedonics; some studies in which already published [1] (under the title of 'Hedonical Calculus'—the species being designated by the generic title) are reprinted here by the kind permission of the Editor of 'Mind.'

Of the Utilitarian Calculus (pp. 56–82) the central conception is *Greatest Happiness*, the greatest possible sum-total of pleasure summed through all time and over all sentience. Mathematical reasonings are employed partly to confirm Mr. Sidgwick's proof that Greatest Happiness is the *end* of right action; partly to deduce middle axioms, *means* conducive to that end. This deduction is of a very abstract, perhaps only negative, character; negativing the assumption that *Equality* is necessarily implied in Utilitarianism. For, if sentients differ in *Capacity for happiness*—under similar circumstances some classes of sentients experiencing on an average more pleasure (*e.g.* of imagination and sympathy) and less pain (*e.g.* of fatigue) than others—there is no presumption that equality of circumstances is the most felicific arrangement; especially when account is taken of the interests of posterity.

Such are the principal topics handled in this *essay* or *tentative* study. Many of the topics, tersely treated in the main body of the work, are more fully illustrated in the course of seven supplementary chapters, or APPENDICES, entitled:

[1] *Mind*, July 1879.

Discussions too much broken up by this arrangement are re-united by references to the principal headings, in the Index; which also refers to the definitions of terms used in a technical sense. The Index also contains the names of many eminent men whose theories, bearing upon the subject, have been noticed in the course of these pages. Dissent has often been expressed. In so terse a composition it has not been possible always to express, what has always been felt, the deference due to the men and the diffidence proper to the subject.

MATHEMATICAL PSYCHICS.

ON THE APPLICATION OF MATHEMATICS TO THE MORAL SCIENCES.

The application of mathematics to *Belief*, the calculus of Probabilities, has been treated by many distinguished writers; the calculus of *Feeling*, of Pleasure and Pain, is the less familiar, but not in reality[1] more paradoxical subject of this essay.

The subject divides itself into two parts; concerned respectively with principle and practice, root and fruit, the applicability and the application of Mathematics to Sociology.

PART I.

In the first part it is attempted to prove an affinity between the moral and the admittedly mathematical sciences from their resemblance as to (1) a certain general complexion, (2) a particular salient feature.

(1) The science of quantity is not alien to the study of man, it will be generally admitted, in so far as actions and effective desires can be *numerically* measured by way of statistics—that is, very far, as Professor Jevons[2] anticipates. But in so far as our *data* may consist of

[1] Cf. Jevons, *Theory*, p. 9.
[2] Introduction to *Theory of Political Economy*.

estimates other than *numerical*, observations that some conditions are accompanied with *greater* or *less* pleasure than others, it is necessary to realise that mathematical reasoning is not, as commonly[1] supposed, limited to subjects where numerical data are attainable. Where there are data which, though not *numerical* are *quantitative*—for example, that a quantity is *greater* or *less* than another, *increases* or *decreases*, is *positive* or *negative*, a *maximum* or *minimum*, there mathematical reasoning is possible and may be indispensable. To take a trivial instance: a is greater than b, and b is greater than c, therefore a is greater than c. Here is mathematical reasoning applicable to quantities which may not be susceptible of numerical evaluation. The following instance is less trivial, analogous indeed to an important social problem. It is required to distribute a given quantity[2] of fuel, so as to obtain the greatest possible quantity of available energy, among a given set of engines, which differ in efficiency—*efficiency* being thus defined: one engine is more efficient than another if, whenever the total quantity of fuel consumed by the former is equal to that consumed by the latter, the total quantity of energy yielded by the former is greater than that yielded by the latter.

In the distribution, shall a larger portion of fuel be given to the more efficient engines? always, or only in some cases? and, if so, in what sort of cases? Here is a very simple problem involving no numerical data, yet

[1] The popular view pervades much of what Mill (in his *Logic*), after Comte, says about Mathematics applied to Sociology. There is a good expression of this view in the *Saturday Review* (on Professor Jevons's *Theory*, November 11, 1871.) The view adopted in these pages is expressed by Cournot, *Recherches*.)

[2] *Or*, a given quantity *per unit of time*, with corresponding modification of definition and problem.

requiring, it may be safely said, mathematics for its complete investigation.

The latter statement may be disputed in so far as such questions may be solved by reasoning, which, though not symbolical, is strictly mathematical; answered more informally, yet correctly, by undisciplined common sense. But, firstly, the advocate of mathematical reasoning in social science is not concerned to deny that mathematical reasoning in social, as well as in physical, science may be divested of symbol. Only it must be remembered that the question how far mathematics can with safety or propriety be divested of her peculiar costume is a very delicate question, only to be decided by the authority and in the presence of Mathematics herself. And, secondly, as to the sufficiency of common sense, the worst of such unsymbolic, at least unmethodic, calculations as we meet in popular economics is that they are apt to miss the characteristic advantages of deductive reasoning. He that will not verify his conclusions as far as possible by mathematics, as it were bringing the ingots of common sense to be assayed and coined at the mint of the sovereign science, will hardly realize the full value of what he holds, will want a measure of what it will be worth in however slightly altered circumstances, a means of conveying and making it current. When the given conditions are not sufficient to determinate the problem —a case of great importance in Political Economy— the ἀγεωμέτρητος is less likely to suspect this deficiency, less competent to correct it by indicating what conditions are necessary and sufficient. All this is evident at a glance through the instrument of mathematics, but to the naked eye of common sense partially and obscurely, and, as Plato says of unscientific knowledge, in a state between genuine Being and Not-Being.

The preceding problem, to distribute a given quantity of material in order to a maximum of energy, with its starting point *loose quantitative relations* rather than numerical data—its slippery though short path almost necessitating the support of mathematics—illustrates fairly well the problem of utilitarian distribution.[1] To illustrate the economical problem of exchange, the maze of many dealers contracting and competing with each other, it is possible to imagine[2] a mechanism of many parts where the law of motion, which particular part moves off with which, is not precisely given—with symbols, arbitrary functions, representing not merely *not numerical knowledge* but[3] *ignorance*—where, though the mode of motion towards equilibrium is indeterminate, the position of equilibrium is mathematically determined.

Examples not made to order, taken from the common stock of mathematical physics, will of course not fit so exactly. But they may be found in abundance, it is submitted, illustrating the property under consideration —mathematical reasoning without numerical data. In Hydrodynamics, for instance, we have a Thomson or Tait[4] reasoning 'principles' for 'determining P and Q *will be given later.* In the meantime it is obvious that *each decreases as X increases.* Hence the equations of motion show '—and he goes on to draw a conclusion of

[1] See p. 64. [2] See p. 34.

[3] *Ignoration of Co-ordinates* (Thomson and Tait, *Natural Philosophy*, 2nd edition), is appropriate in many social problems where we only know in part.

[4] Thomson and Tait, *Treatise on Natural Philosophy*, p. 320, 2nd edition. The italics, which are ours, call attention to the *unnumerical, loose quantitative, relation* which constitutes the datum of the mathematical reasoning.

momentous interest that balls (properly) projected in an infinite incompressible fluid will move as if they were attracted to each other. And generally in the higher Hydrodynamics, in that boundless ocean of perfect fluid, swum through by vortices, where the deep first principles of Physics are to be sought, is not a similar *unnumerical*, or *hyperarithmetical* method there pursued? If a portion of perfect fluid so moves at any time that each particle has no motion of rotation, then that portion of the fluid will retain that property for all time[1]; here is no application of the numerical measuring-rod.

No doubt it may be objected that these hydrodynamical problems employ some *precise* data; the very definition of Force, the conditions of fluidity and continuity. But so also have our social problems *some* precise data: for example, the property of *uniformity of price* in a market; or rather the (approximately realised) conditions of which that property is the deducible *effect*, and which bears a striking resemblance to the data of hydrodynamics:[2] (1) the *fulness* of the market: that there *continues* to be up to the conclusion of the dealing an indefinite number of dealers; (2) the *fluidity* of the market, or infinite dividedness of the dealers' interests. Given this property of uniform price, Mr. Marshall and M. Walras deduce mathematically, though not arithmetically, an interesting theorem, which Mill and Thornton failed with unaided reason to discern, though they were quite close to it—the theorem that the equation of supply to demand, though a necessary, is not a sufficient condition of market price.

To attempt to select representative instances from each

[1] Stokes, *Mathematical Papers*, p. 112.

[2] See p. 18.

recognised branch of mathematical inquiry would exceed the limits of this paper and the requirements of the argument. It must suffice, in conclusion, to direct attention to one species of Mathematics which seems largely affected with the property under consideration, the Calculus of Maxima and Minima, or (in a wide sense) *of Variations*. The criterion of a *maximum*[1] turns, not upon the *amount*, but upon the *sign* of a certain quantity.[2] We are continually concerned[3] with the ascertainment of a certain *loose quantitative relation*, the *decrease-of-rate-of-increase* of a quantity. Now, this is the very quantitative relation which it is proposed to employ in mathematical sociology; given in such data as the *law of diminishing returns to capital and labour*, the *law of diminishing utility*, the *law of increasing fatigue*; the very same irregular, unsquared material which constitutes the basis of the Economical and the Utilitarian Calculus.

Now, it is remarkable that the principal inquiries in Social Science may be viewed as *maximum-problems*. For Economics investigates the arrangements between agents each tending to his own *maximum* utility; and Politics and (Utilitarian) Ethics investigate the arrangements which conduce to the *maximum* sum total of

[1] *Maximum* in this paper is employed according to the context for (1) *Maximum* in the proper mathematical sense; (2) *Greatest possible*; (3) *stationary*; (4) where *minimum* (or *least possible*) might have been expected; upon the principle that every minimum is the correlative of a maximum. Thus Thomson's Minimum theorem is correlated with Bertrand's Maximum theorem. (Watson and Burbury.) This liberty is taken, not only for brevity, but also for the sake of a certain suggestiveness. '*Stationary*,' for instance, fails to suggest the *superlativeness* which it connotes.

[2] The second term of Variation. It may be objected that the *other* condition of a maximum equation of the first term to zero is of a more *precise* character. See, however, Appendix I., p. 92.

[3] E.g., Todhunter's *Researches on Calculus of Variations*, pp. 21–30, 80, 117, 286, &c.

utility. Since, then, Social Science, as compared with the Calculus of Variations, starts from similar data—*loose quantitative relations*—and travels to a similar conclusion—determination of *maximum*—why should it not pursue the same method, Mathematics?

There remains the objection that in Physical Calculus there is always (as in the example quoted above from Thomson and Tait) a potentiality, an expectation, of measurement; while Psychics want the first condition of calculation, *a unit*. The following[1] brief answer is diffidently offered.

Utility, as Professor Jevons[2] says, has two dimensions, *intensity* and *time*. The unit in each dimension is the just perceivable[3] increment. The implied equation to each other of each *minimum sensibile* is a first principle incapable of proof. It resembles the equation to each other of undistinguishable events or cases,[4] which constitutes the first principle of the mathematical calculus of *belief*. It is doubtless a principle acquired in the course of evolution. The implied equatability of time-intensity units, irrespective of distance in time and kind of pleasure, is still imperfectly evolved. Such is the unit of *economical* calculus.

For moral calculus a further dimension is required; to compare the happiness of one person with the happiness of another, and generally the happiness of groups of different members and different average happiness.

Such comparison can no longer be shirked, if there

[1] For a fuller discussion, see Appendix III.

[2] In reference to Economics, *Theory*, p. 51.

[3] Cf. Wundt, *Physiological Psychology*; below, p. 60. Our 'ebenmerklich' minim is to be regarded not as an infinitesimal differential, but as a finite small difference; a conception which is consistent with a (duly cautious) employment of infinitesimal notation.

[4] Laplace, *Essai—Probabilities*, p. 7.